Lecture 29

Uniqueness Theorem

The uniqueness of a solution to a linear system of equations is an important concept in mathematics. Under certain conditions, ordinary differential equation partial differential equation and matrix equations will have unique solutions under the prescribed boundary condition and the driving source terms. This is the manner of how we solve a boundary value problem. But uniqueness of a boundary value problem is not always guaranteed as we shall see. This issue is discussed in many math books and linear algebra books [79, 92]. The proof of uniqueness for Laplace and Poisson equations are given in [32, 55] which is slightly different from electrodynamic problems.

Just imagine how bizzare it would be if there are more than one possible solutions. One has to determine which is the real solution. To quote Star Trek, we need to know who the real McCoy is![1]

29.1 The Difference Solutions to Source-Free Maxwell's Equations

In this section, we will prove uniqueness theorem for electrodynamic problems under the prescribed boundary condition with unique sources in the system [33, 36, 51, 66, 85]. This is important, as when we solve Maxwell's equations, we are solving a set of partial differential equations as a boundary value problem with prescribed boundary conditions. We like to know when such a problem has a unique solution.

First, let us assume that there exist two solutions in the presence of one set of common impressed sources $\mathbf{J}_i$ and $\mathbf{M}_i$.[2] Namely, these two solutions are $\mathbf{E}^a$, $\mathbf{H}^a$, $\mathbf{E}^b$, $\mathbf{H}^b$. Both of them satisfy Maxwell's equations and the same boundary conditions. Are $\mathbf{E}^a = \mathbf{E}^b$, $\mathbf{H}^a = \mathbf{H}^b$?

[1]This phrase was made popular to the baby-boom generation, or the Trekkies by Star Trek. It actually refers to an African American inventor.

[2]It is not clear when the useful concept of impressed sources were first used in electromagnetics even though it was used in [187] in 1936. These are immutable sources that cannot be changed by the environment in which they are immersed.

To study the uniqueness theorem, we consider general linear anisotropic inhomogeneous media, where the tensors $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ can be complex so that lossy media can be included. In the frequency domain, lets assume two possible solutions with one given set of sources $\mathbf{J}_i$ and $\mathbf{M}_i$, it follows that

$$\nabla \times \mathbf{E}^a = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^a - \mathbf{M}_i \tag{29.1.1}$$

$$\nabla \times \mathbf{E}^b = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^b - \mathbf{M}_i \tag{29.1.2}$$

$$\nabla \times \mathbf{H}^a = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^a + \mathbf{J}_i \tag{29.1.3}$$

$$\nabla \times \mathbf{H}^b = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^b + \mathbf{J}_i \tag{29.1.4}$$

By taking the difference of these two solutions, we have

$$\nabla \times (\mathbf{E}^a - \mathbf{E}^b) = -j\omega\overline{\boldsymbol{\mu}} \cdot (\mathbf{H}^a - \mathbf{H}^b) \tag{29.1.5}$$

$$\nabla \times (\mathbf{H}^a - \mathbf{H}^b) = j\omega\overline{\boldsymbol{\varepsilon}} \cdot (\mathbf{E}^a - \mathbf{E}^b) \tag{29.1.6}$$

Or alternatively, defining $\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b$ and $\delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$, we have

$$\nabla \times \delta\mathbf{E} = -j\omega\overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \tag{29.1.7}$$

$$\nabla \times \delta\mathbf{H} = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \delta\mathbf{E} \tag{29.1.8}$$

The difference solutions, $\delta\mathbf{E}$ and $\delta\mathbf{H}$, satisfy the original source-free Maxwell's equations. Source-free here implies that we are looking at the homogeneous solutions of the pertinent partial differential equations constituted by (29.1.7) and (29.1.8).

To prove uniqueness, we would like to find a simplifying expression for $\nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*)$. By using the product rule for divergence operator, it can be shown that

$$\nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) = \delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* \tag{29.1.9}$$

We need to simplify the right-hand side of the above with the goal of proving the uniqueness theorem. Then by taking the left dot product of $\delta\mathbf{H}^*$ with (29.1.7), and then the left dot product of $\delta\mathbf{E}$ with the complex conjugation of (29.1.8), we obtain

$$\begin{aligned}\delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \\ \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= -j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*\end{aligned} \tag{29.1.10}$$

Now, taking the difference of the above, we get

$$\begin{aligned}\delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) \\ &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*\end{aligned} \tag{29.1.11}$$

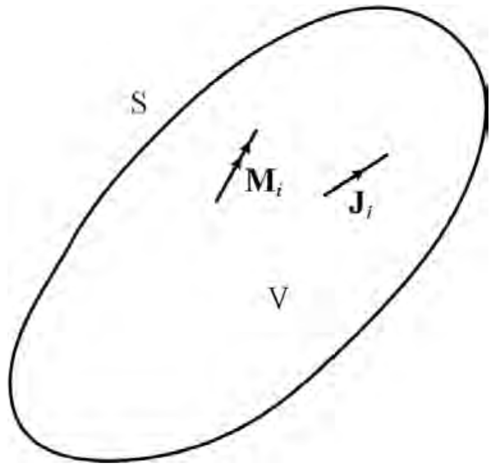

Figure 29.1: Geometry for proving the uniqueness theorem. We like to know the requisite boundary conditions on S plus the type of media inside V in order to guarantee the uniqueness of the solution in V.

Our goal is to find the conditions under which $\delta\mathbf{H}$ and $\delta\mathbf{E}$ are both zero, which will guarantee uniqueness of the solution. Next, we integrate the above equation (29.1.11) over a volume V bounded by a surface S as shown in Figure 29.1. After making use of Gauss' divergence theorem, we arrive at

$$\iint_V \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) dV = \oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S} \tag{29.1.12}$$

or that

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S} = \iiint_V [-j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV \tag{29.1.13}$$

And next, we would like to know the kind of boundary conditions that would make the left-hand side equal to zero. It is seen that the surface integral on the left-hand side will be zero if:[3]

1. If $\hat{n} \times \mathbf{E}$ is specified over S for the two possible solutions, so that $\hat{n} \times \mathbf{E}_a = \hat{n} \times \mathbf{E}_b$ on S. Then $\hat{n} \times \delta\mathbf{E} = 0$, which is the PEC boundary condition for $\delta\mathbf{E}$, and then[4]
$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = \oiint_S (\hat{n} \times \delta\mathbf{E}) \cdot \delta\mathbf{H}^* dS = 0.$

2. If $\hat{n} \times \mathbf{H}$ is specified over S for the two possible solutions, so that $\hat{n} \times \mathbf{H}_a = \hat{n} \times \mathbf{H}_b$ on S. Then $\hat{n} \times \delta\mathbf{H} = 0$, which is the PMC boundary condition for $\delta\mathbf{H}$, and then

[3] In the following, please be reminded that PEC stands for "perfect electric conductor", while PMC stands for "perfect magnetic conductor". PMC is the dual of PEC. Also, a fourth case of impedance boundary condition is possible, which is beyond the scope of this course. Interested readers may consult Chew, Theory of Microwave and Optical Waveguides [85].

[4] We use the vector identity that $\mathbf{a} \cdot (\mathbf{b} \times c) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$. Also, from Section 1.3.3, $d\mathbf{S} = \hat{n} dS$.

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = -\oiint_S (\hat{n} \times \delta\mathbf{H}^*) \cdot \delta\mathbf{E} dS = 0.$$

3. Let the surface S be divided into two mutually exclusive surfaces S_1 and S_2.[5] If $\hat{n} \times \mathbf{E}$ is specified over S_1, and $\hat{n} \times \mathbf{H}$ is specified over S_2. Then $\hat{n} \times \delta\mathbf{E} = 0$ (PEC boundary condition) on S_1, and $\hat{n} \times \delta\mathbf{H} = 0$ (PMC boundary condition) on S_2. Therefore, the left-hand side becomes

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = \iint_{S_1} + \iint_{S_2} = \iint_{S_1} (\hat{n} \times \delta\mathbf{E}) \cdot \delta\mathbf{H}^* dS - \iint_{S_2} (\hat{n} \times \delta\mathbf{H}^*) \cdot \delta\mathbf{E} dS = 0.$$

Thus, under the above three scenarios, the left-hand side of (29.1.13) is zero, and then the right-hand side of (29.1.13) becomes

$$\iiint_V [-j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \delta\mathbf{E}^*] dV = 0 \tag{29.1.14}$$

For lossless media, $\overline{\boldsymbol{\mu}}$ and $\overline{\varepsilon}$ are hermitian tensors (or matrices[6]), then it can be seen, using the properties of hermitian matrices or tensors, that $\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}$ and $\delta\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \delta\mathbf{E}^*$ are purely real. Taking the imaginary part of the above equation yields

$$\iiint_V [-\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + \delta\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \delta\mathbf{E}^*] dV = 0 \tag{29.1.15}$$

The above two terms correspond to stored magnetic field energy and stored electric field energy in the difference solutions $\delta\mathbf{H}$ and $\delta\mathbf{E}$, respectively. The above being zero does not imply that $\delta\mathbf{H}$ and $\delta\mathbf{E}$ are zero since they can be negative of each other.

For resonant solutions, the stored electric energy can balance the stored magnetic energy. The above resonant solutions are those of the difference solutions satisfying PEC or PMC boundary condition or mixture thereof. Also, they are the resonant solutions of the source-free Maxwell's equations (29.1.7). Therefore, $\delta\mathbf{H}$ and $\delta\mathbf{E}$ need not be zero, even though (29.1.15) is zero. This happens when we encounter solutions that are the resonant modes of the volume V bounded by the surface S.

29.2 Conditions for Uniqueness

Uniqueness can only be guaranteed if the medium is lossy as shall be shown later. It is also guaranteed if lossy impedance boundary conditions are imposed.[7] First we begin with the isotropic case.

29.2.1 Isotropic Case

It is easier to see this for lossy isotropic media. Then (29.1.14) simplifies to

$$\iiint_V [-j\omega\mu|\delta\mathbf{H}|^2 + j\omega\varepsilon^*|\delta\mathbf{E}|^2] dV = 0 \tag{29.2.1}$$

[5]In math parlance, $S_1 \cup S_2 = S$.

[6]Tensors are a special kind of matrices.

[7]See Chew, Theory of Microwave and Optical Waveguides.

For isotropic lossy media, $\mu = \mu' - j\mu''$ and $\varepsilon = \varepsilon' - j\varepsilon''$. Taking the real part of the above, we have from (29.2.1) that

$$\iiint_V [-\omega\mu''|\delta\mathbf{H}|^2 - \omega\varepsilon''|\delta\mathbf{E}|^2]dV = 0 \tag{29.2.2}$$

Since the integrand in the above is always negative definite, the integral can be zero only if

$$\delta\mathbf{E} = 0, \quad \delta\mathbf{H} = 0 \tag{29.2.3}$$

everywhere in V, implying that $\mathbf{E}_a = \mathbf{E}_b$, and that $\mathbf{H}_a = \mathbf{H}_b$. Hence, it is seen that uniqueness is guaranteed only if the medium is lossy.

The physical reason is that when the medium is lossy, a homogeneous solution (also called a natural solution) which is pure time-harmonic solution cannot exist due to loss. The modes, which are the source-free solutions of Maxwell's equations, are decaying sinusoids. But when we express equations (29.1.1) to (29.1.4) in the frequency domain, we are seeking solutions for which ω is real. Thus decaying sinusoids are not among the possible solutions, and hence, they are not in the solution space.

Notice that the same conclusion can be drawn if we make μ'' and ε'' negative. This corresponds to active media, and uniqueness can be guaranteed for a time-harmonic solution. In this case, no time-harmonic solution exists, and the resonant solution is a growing sinusoid. Therefore, uniqueness is guaranteed for active or passive media. However, if the medium is a mixed of active and passive media, uniqueness is not guaranteed again.

29.2.2 General Anisotropic Case

The proof for general anisotropic media is more complicated. For the lossless anisotropic media, we see that (29.1.14) is purely imaginary. However, when the medium is lossy, this same equation will have a real part. Hence, we need to find the real part of (29.1.14) for the general lossy case.

About taking the Real and Imaginary Parts of a Complicated Expression

To this end, we digress on taking the real and imaginary parts of a complicated expression. Here, we need to find the complex conjugate[8] of (29.1.14), which is scalar, and add it to itself to get its real part. To this end, we will find the conjugate of its integrand which is a scalar number.

First, the complex conjugate of the first scalar term in the integrand of (29.1.14) is[9]

$$(-j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H})^* = j\omega\delta\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \delta\mathbf{H}^* = j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \delta\mathbf{H} \tag{29.2.4}$$

Similarly, the complex conjugate of the second scalar term in the same integrand is

$$(j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*)^* = -j\omega\delta\mathbf{E}^* \cdot \overline{\boldsymbol{\varepsilon}}^\dagger \cdot \delta\mathbf{E} \tag{29.2.5}$$

[8] Also called hermitian conjugate.

[9] To arrive at these expressions, one makes use of the matrix algebra rule that if $\overline{\mathbf{D}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}}$, then $\overline{\mathbf{D}}^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$. This is true even for non-square matrices. But for our case here, $\overline{\mathbf{A}}$ is a 1×3 row vector, and $\overline{\mathbf{C}}$ is a 3×1 column vector, and $\overline{\mathbf{B}}$ is a 3×3 matrix. In vector algebra, the transpose of a vector is implied. Also, in our case here, $\overline{\mathbf{D}}$ is a scalar, and hence, its transpose is itself.

But

$$j\omega\delta\mathbf{E}\cdot\overline{\varepsilon}^*\cdot\delta\mathbf{E}^* = j\omega\delta\mathbf{E}^*\cdot\overline{\varepsilon}^\dagger\cdot\delta\mathbf{E} \tag{29.2.6}$$

The above gives us the complex conjugate of the scalar quantity (29.1.14) and adding it to itself, we have

$$\iiint_V [-j\omega\delta\mathbf{H}^*\cdot(\overline{\boldsymbol{\mu}}-\overline{\boldsymbol{\mu}}^\dagger)\cdot\delta\mathbf{H} - j\omega\delta\mathbf{E}^*\cdot(\overline{\varepsilon}-\overline{\varepsilon}^\dagger)\cdot\delta\mathbf{E}]dV = 0 \tag{29.2.7}$$

For lossy media, $-j(\overline{\boldsymbol{\mu}}-\overline{\boldsymbol{\mu}}^\dagger)$ and $-j(\overline{\varepsilon}-\overline{\varepsilon}^\dagger)$ are hermitian positive matrices. Hence the integrand is always positive definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V. Thus, uniqueness is guaranteed in a lossy anisotropic medium.

Similar statement can be made for the isotropic case if the medium is active. Then the integrand is positive definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V, thereby proving that uniqueness is satisfied.

29.3 Hind Sight Using Linear Algebra

The proof of uniqueness for Maxwell's equations is very similar to the proof of uniqueness for a matrix equation [79]. As you will see, the proof using linear algebra is a lot simpler due to the simplicity of notations. To see this, consider a linear algebraic equation

$$\overline{\mathbf{A}}\cdot\mathbf{x} = \mathbf{b} \tag{29.3.1}$$

If a solution to a matrix equation exists without excitation, namely, when $\mathbf{b} = 0$, then the solution is the null space solution [79], namely, $\mathbf{x} = \mathbf{x}_N$. In other words,

$$\overline{\mathbf{A}}\cdot\mathbf{x}_N = 0 \tag{29.3.2}$$

These null space solutions are solutions that exist without a "driving term" $\mathbf{b}$ on the right-hand side. In the parlance of linear algebra, such a matrix system does not have a unique solution, or the matrix inverse does not exist, or the matrix system is singular. This is very different from a matrix operator as a linear map.

For Maxwell's equations, $\mathbf{b}$ corresponds to the source terms. The solution in (29.3.2) is like the homogeneous solution of an ordinary differential equation or a partial differential equation [92]. In an enclosed region of volume V bounded by a surface S, homogeneous solutions are the resonant solutions (or the natural solutions) of this Maxwellian system. When these solutions exist, they give rise to non-uniqueness. Note that these resonant solutions in the time domain exist for all time if the cavity is lossless.

Also, notice that (29.1.7) and (29.1.8) are Maxwell's equations without the source terms. In a closed region V bounded by a surface S, only resonant solutions for $\delta\mathbf{E}$ and $\delta\mathbf{H}$ with the relevant boundary conditions can exist when there are no source terms.

As previously mentioned, one way to ensure that these resonant solutions (or homogeneous solutions) are eliminated is to put in loss or gain. When loss or gain is present, then the resonant solutions are decaying sinusoids or growing sinusoids (see Section 22.1.1 for an analogue with LC tank circuit). Since we are looking for solutions in the frequency domain, or time harmonic solutions, the solutions we are seeking are on the real ω axis on the complex ω plane. Thus the non-sinusoidal solutions are outside the solution space: They are not part of the time-harmonic solutions (which are on the real axis) that we are looking for. Therefore, complex resonant solutions which are off the real axis, and are homogeneous solutions, are not found on the real axis.

We see that the source of non-uniqueness is the homogeneous solutions or the resonant solutions of the system that persist for all time. These solutions are non-causal, and they are there in the system since the beginning of time to time tending to infinity or *ad infinitum*. One way to remove these resonant solutions is to set them to zero at the beginning by solving an initial value problem (IVP). However, this has to be done in the time domain. Hence, one reason for non-uniqueness is because we are seeking the solutions in the frequency domain.

29.4 Connection to Poles of a Linear System

The output is the response to the input of a linear system. It can be represented by a transfer function $H(\omega)$ [53, 188]. If $H(\omega)$ has poles, and if the system is lossless, the poles are on the real axis. Therefore, when $\omega = \omega_{\text{pole}}$, the function $H(\omega)$ becomes undefined. In other words, one can add a constant term to the output, and the ratio between output to input is still infinity. This also is the reason for non-uniqueness of the output with respect to the input. Poles usually correspond to resonant solutions, and hence, the non-uniqueness of the solution is intimately related to the non-uniqueness of Maxwell's equations at the resonant frequencies of a structure. This is illustrated in the upper part of Figure 29.2.

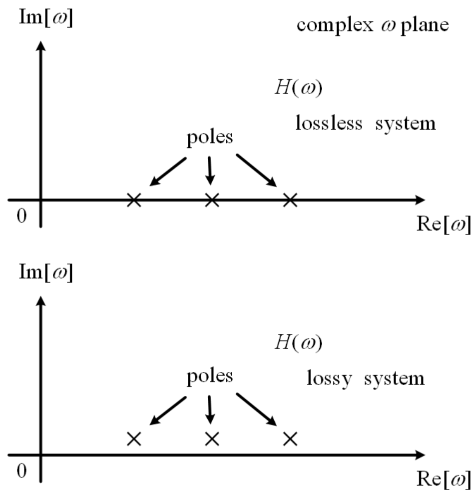

Figure 29.2: The non-uniqueness problem is intimately related to the locations of the poles of a transfer function being on the real axis, when one solves a linear system using Fourier transform technique. For a lossless system, the poles are located on the real axis. When performing a Fourier inverse transform to obtain the solution in the time domain, then the Fourier inversion contour is undefined, and the solution cannot be uniquely determined.

If the input function is $f(t)$, with Fourier transform $F(\omega)$, then the output $y(t)$ is given by the following Fourier integral, viz.,

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{j\omega t} H(\omega) F(\omega) \tag{29.4.1}$$

where the Fourier inversion integral path is on the real axis on the complex ω plane. The Fourier inversion integral above is undefined or non-unique if poles exist on the real ω axis.

However, if loss is introduced, these poles will move away from the real axis as shown in the lower part of Figure 29.2. Then the transfer function is uniquely determined for all frequencies on the real axis. In this way, the Fourier inversion integral in (29.4.1) is well defined, and uniqueness of the solution is guaranteed.

When the poles are located on the real axis yielding possibly non-unique solutions, a remedy to this problem is to use Laplace transform technique [53]. The Laplace transform technique allows the specification of initial values, which is similar to solving the problem as an initial value problem (IVP). As mentioned before, solving these problems as an IVP will remove non-uniqueness of the solution.

If you have problem wrapping your head around this concept, it is good to connect back to the LC tank circuit example. The transfer function $H(\omega)$ is similar to the $Y(\omega)$ of (22.1.4). The transfer function has two poles. If there is no loss, then the poles are located on the real axis, rendering the Fourier inversion contour undefined in (29.4.1). Hence, the solution is non-unique. However, if infinitesimal loss is introduced by setting $R \neq 0$, then the poles will migrate off the real axis making (29.4.1) well defined!

29.5 Radiation from Antenna Sources and Radiation Condition

The above uniqueness theorem guarantees that if we have some antennas with prescribed current sources on them, the radiated field from these antennas are unique under certain conditions. To see how this can come about, we first study the radiation of sources into a region V bounded by a large surface S_{inf} as shown in Figure 29.4 [36].

Even when $\hat{n} \times \mathbf{E}$ or $\hat{n} \times \mathbf{H}$ are specified on the surface at S_{inf}, the solution is nonunique because the volume V bounded by S_{inf}, can have many resonant solutions. In fact, the region will be replete with resonant solutions as one makes S_{inf} become very large.

To gain more insight, we look at the resonant condition of a large rectangular cavity given by (21.2.3) reproduced here as

$$\beta^2 = \frac{\omega^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \tag{29.5.1}$$

The above is an equation of an Ewald sphere in a 3D mode space which is described by discrete points, or that the values of $\beta_x = \frac{m\pi}{a}$, $\beta_y = \frac{n\pi}{b}$, and $\beta_z = \frac{p\pi}{d}$ are discrete. We can continuously change the operating frequency ω above until the above equation is satisfied. When this happens, we encounter a resonant frequency of the cavity. At this operating frequency, the solution to Maxwell's equations inside the cavity is non-unique. As the dimensions of the cavity become large or a, b, and d are large, then the number of ω's or resonant frequencies that the above equation can be satisfied or approximately satisfied becomes very large. This is illustrated Figure 29.3 in 2D. Hence, the chance of the operating frequency ω coinciding with a resonant mode of the cavity is very high giving rise to non-uniqueness. This is even more so when the cavity becomes very large. Hence, the chance of operating inside the large cavity with unique solution is increasingly small. This above argument applies to cavities of other shapes as well.

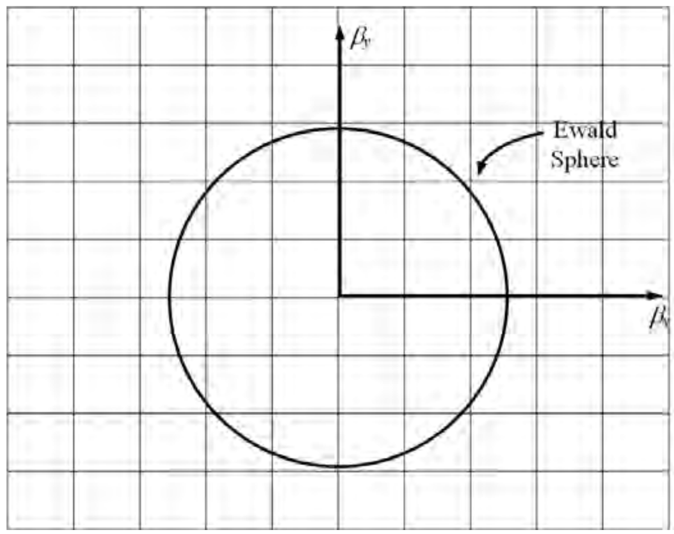

Figure 29.3: For very large cavity, the grid spacing in the mode space (or Fourier space) becomes very small. Then the chance that the sphere surface encounters a resonant mode is very high. When this happens, the solution to the cavity problem is non-unique.

The way to remove these resonant solutions is to introduce an infinitesimal amount of loss in region V. Then these resonant solutions will disappear from the real ω axis, where we seek a time-harmonic solution. Now we can take S_{inf} to infinity, and the solution will always be unique even if the loss is infinitesimally small.

Notice that if $S_{\text{inf}} \to \infty$, the waves that leave the sources will never be reflected back because of the small amount of loss. The radiated field will just disappear into infinity. This is just what radiation loss is: power that propagates to infinity, but never to return. In fact, one way of guaranteeing the uniqueness of the solution in region V when S_{inf} is infinitely large, or that V is infinitely large is to impose the radiation condition: the waves that radiate to infinity are outgoing waves only, and never do they return. This is also called the *Sommerfeld radiation condition* [189]. . Uniqueness of the field outside the sources is always guaranteed if we assume that the field radiates to infinity and never to return. This is equivalent to solving the cavity solutions with an infinitesimal loss, and then letting the size of the cavity become infinitely large.

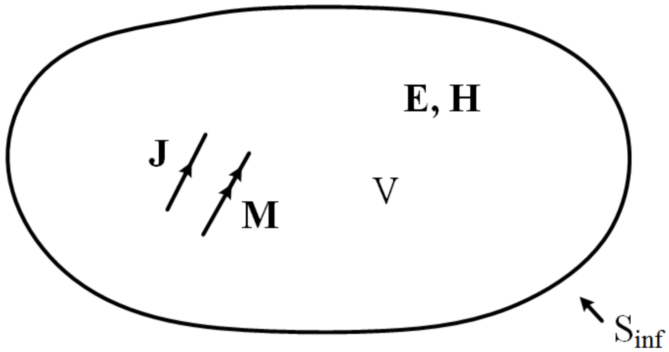

Figure 29.4: The solution for antenna radiation is unique because we impose the Sommerfeld radiation condition when seeking the solution. That is we assume that the radiation wave travels to infinity but never to return. This is equivalent to assuming an infinitesimal loss when seeking the solution in V and later let $V \to \infty$.